

Entanglement of Formation of an Arbitrary State of Two Rebits*

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Abstract

We consider entanglement for quantum states defined in vector spaces over the real numbers. Such real entanglement is different from entanglement in standard quantum mechanics over the complex numbers. The differences provide insight into the nature of entanglement in standard quantum theory. Wootters [Phys. Rev. Lett. **80**, 2245 (1998)] has given an explicit formula for the entanglement of formation of two qubits in terms of what he calls the *concurrence* of the joint density operator. We give a contrasting formula for the entanglement of formation of an arbitrary state of two “rebits,” a rebit being a system whose Hilbert space is a 2-dimensional real vector space.

I. INTRODUCTION

One of the key distinguishing features of quantum mechanics, not found in classical physics, is the possibility of entanglement between subsystems. The significance of this phenomenon is now unquestioned, as it lies at the core of several of the most important achievements of quantum information science [1], such as quantum teleportation [2] and quantum error correction [3,4]. Yet can we say that we understand the distinction between those physical theories with entanglement and those without? It is difficult to claim such understanding, as our most well studied foil theory to date—namely, classical physics—is completely devoid of the phenomenon. This paper, in a small way, contributes to filling that

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gap by studying entanglement in real vector spaces, where though there is entanglement, it is different from the entanglement in the complex vector spaces of standard quantum theory.

Our specific objective here is to analyze entanglement for states of two “rebits,” a rebit being a two-state system whose Hilbert space is defined over the field of real numbers. In particular, we give an explicit formula for the entanglement of formation of an arbitrary (mixed) state of two rebits. This formula is similar in some structural ways, but not identical to Wootters’s formula for the entanglement of formation of two qubits [5]. The reason for our considering real vector spaces is just the reason described above: they provide an easy, well defined foil with which to compare standard quantum theory; indeed, they have already been fruitful in that regard [6]. A quantum theory that uses real vector spaces is similar to, but not identical to the standard theory [7,8]. In general, we follow the philosophy of Weinberg in Ref. [9] in the hope that this exercise will help identify those aspects of entanglement that are unique, those that are accidental, and those that are necessary to the standard theory.

To state our main result, we must build some concepts and notation. Consider a bipartite composite system, made up of subsystems A and B . A density operator ρ^{AB} of the composite system, pure or mixed, is said to be *separable* if it can be thought of as arising from an ensemble of product states, i.e.,

$$\rho^{AB} = \sum_j p_j \rho_j^A \otimes \rho_j^B . \quad (1.1)$$

A separable pure state is itself a product state. The reason this definition is interesting is because a separable state can be created by procedures that are local to each subsystem, whereas a nonseparable state cannot be created by any local means.

Taking the matrix transpose of any density operator relative to some orthonormal basis—this is the same as taking the complex conjugate in that basis—yields another density operator, i.e., another positive semi-definite operator with unit trace. Similarly, if a state of a bipartite system is separable, taking the *partial transpose* on system B in any basis also yields another density operator. If, however, taking the partial transpose leads to an operator that is not positive semi-definite, one can be sure that the original state was an entangled state. This is the partial transpose condition of Peres [10]. Unfortunately, for subsystems A and B of arbitrary Hilbert-space dimensions, the converse of the Peres condition is not true—the partial transpose of an entangled state can give another positive semi-definite operator. Thus the Peres condition does not provide a general criterion for testing entanglement. For 2×2 systems (two qubits) or 2×3 systems (a qubit and a qutrit), however, the Peres condition does provide a criterion for entanglement: a state of such a composite system is entangled if and only if its partial transpose is a nonpositive operator, i.e., has at least one negative eigenvalue [11].

The chief resource-based measures of entanglement are the *entanglement of formation* and the *distillable entanglement* [12,13]. For two d -dimensional systems, the pure state

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^d |e_j^A\rangle \otimes |e_j^B\rangle , \quad (1.2)$$

where $|e_j^A\rangle$ and $|e_j^B\rangle$ are orthonormal bases for the two subsystems, is maximally entangled in the sense that it can be used to teleport the state of another d -dimensional system. The degree of entanglement of such a maximally entangled state is $\log_2 d$, the marginal entropy of

each subsystem. Suppose that starting with m such maximally entangled states, one has a procedure, involving only local operations and classical communication between subsystems, for creating n copies of an arbitrary state ρ^{AB} . The entanglement of formation, $E(\rho^{AB})$, is defined to be $\log_2 d$ times the asymptotic ratio m/n for an optimal procedure, i.e., one that has the smallest such ratio. Similarly, suppose that starting with n copies of the state ρ^{AB} , one has a procedure, again involving only local operations and classical communication, for distilling m maximally entangled states. The distillable entanglement, $D(\rho^{AB})$, is defined to be $\log_2 d$ times the asymptotic ratio m/n for an optimal procedure, i.e., one that has the largest such ratio.

A separable state has no entanglement of either sort, whereas a nonseparable state necessarily has a nonzero entanglement of formation. For pure states, the formation process is reversible, so the entanglement of formation and the distillable entanglement are the same. For mixed states, however, the distillable entanglement is generally less than the entanglement of formation, reflecting the irreversibility of the formation process. Interestingly, a state with a positive partial transpose has no distillable entanglement [14]. For 2×2 and 2×3 systems, all entangled states have a nonpositive partial transpose, as noted above, and they also have a nonzero distillable entanglement. For 3×3 and higher-dimensional systems, however, there are entangled states that have positive partial transpose; though these states have nonzero entanglement of formation, they have no distillable entanglement. This kind of entanglement, from which no pure-state entanglement can be distilled, is called *bound entanglement* [14].

The entanglement of formation of a pure state $|\Psi\rangle$ of a bipartite system is given by the entropy of the marginal density operators, ρ^A and ρ^B :

$$E(\Psi) = -\text{tr}(\rho^A \log_2 \rho^A) = -\text{tr}(\rho^B \log_2 \rho^B) . \quad (1.3)$$

For a bipartite mixed state the entanglement of formation is more complicated. A mixed state ρ^{AB} has an ensemble decomposition in terms of pure states $|\Psi_j\rangle$, with probabilities p_j , if it can be written as

$$\rho^{AB} = \sum_j p_j |\Psi_j\rangle \langle \Psi_j| . \quad (1.4)$$

Modulo a presently unanswered question about the super-additivity of the entanglement of formation [15], the entanglement of formation of ρ^{AB} is given by the minimum average entanglement of formation of the pure states in an ensemble [13],

$$E(\rho^{AB}) = \min_{\{p_j, |\Psi_j\rangle\}} \sum_j p_j E(\Psi_j) , \quad (1.5)$$

where the minimum is taken over all possible ensemble decompositions. For two qubits, Wootters has given an explicit formula for the entanglement of formation in terms of what he calls the *concurrence* of the joint density operator [5,16] [see Eq. (2.4) for Wootters's concurrence expression]. There are no known explicit formulae for distillable entanglement, even for 2×2 systems. We now have all the facts about entanglement we need for posing the questions addressed in this paper.

Let us turn now to issues relevant for distinguishing the theory of real quantum mechanics from standard complex quantum mechanics. The vector space of operators on a

d -dimensional vector space (real or complex) is the direct sum of two natural subspaces, the space \mathcal{S} of real, symmetric matrices, which has dimension $\frac{1}{2}d(d+1)$, and the space \mathcal{A} of real, antisymmetric matrices, which has dimension $\frac{1}{2}d(d-1)$. If the vector space is over the real numbers, all the quantum states and observables lie in the symmetric subspace. For a complex vector space, the states and observables are described by Hermitian operators; the (real) vector space \mathcal{H} of Hermitian operators takes advantage of both the symmetric and antisymmetric subspaces, it being the direct sum $\mathcal{H} = \mathcal{S} \oplus i\mathcal{A}$.

This is of significance for entanglement for the following reason. Suppose one combines two systems, with dimensions d_A and d_B , to make a composite system with dimension $d_A d_B$. In the complex case, the composite vector space of Hermitian operators, of dimension $d_A^2 d_B^2$, is the tensor product of the corresponding spaces for A and B , i.e., $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. In contrast, the composite space of symmetric matrices, of dimension $\frac{1}{2}d_A d_B (d_A d_B + 1)$, is not just the tensor product of the symmetric spaces of A and B , but rather is given by the direct sum

$$\mathcal{S}_{AB} = (\mathcal{S}_A \otimes \mathcal{S}_B) \oplus (\mathcal{A}_A \otimes \mathcal{A}_B) . \quad (1.6)$$

Joint states that have a component in the doubly antisymmetric space, $\mathcal{A}_A \otimes \mathcal{A}_B$, are necessarily entangled relative to the real vector space, since product states cannot have a component in $\mathcal{A}_A \otimes \mathcal{A}_B$.

Since any operation in the real vector space can be used in the associated complex vector space, an optimal real procedure, either for entanglement of formation E or for distillable entanglement D , is never better than an optimal complex procedure. The consequence is that for a joint state ρ^{AB} in the real vector space, the real (R) and complex (C) entanglement measures satisfy the following inequalities:

$$E_R(\rho^{AB}) \geq E_C(\rho^{AB}) \geq D_C(\rho^{AB}) \geq D_R(\rho^{AB}) . \quad (1.7)$$

These ideas are made concrete by considering rebits. The three-dimensional space of real, symmetric matrices is spanned by the unit matrix I and two Pauli matrices, σ_x and σ_z , whereas the one-dimensional space of real, antisymmetric matrices is spanned by $i\sigma_y$. For two rebits, the nine-dimensional space $\mathcal{S}_A \otimes \mathcal{S}_B$ is spanned by the matrices

$$\begin{array}{ccc} I \otimes I & \sigma_x \otimes I & \sigma_z \otimes I \\ I \otimes \sigma_x & \sigma_x \otimes \sigma_x & \sigma_z \otimes \sigma_x \\ I \otimes \sigma_z & \sigma_x \otimes \sigma_z & \sigma_z \otimes \sigma_z \end{array} , \quad (1.8)$$

but the entire symmetric subspace \mathcal{S}_{AB} includes one additional basis matrix,

$$i\sigma_y \otimes i\sigma_y = -\sigma_y \otimes \sigma_y . \quad (1.9)$$

Any state of the composite system that contains a $\sigma_y \otimes \sigma_y$ component is necessarily entangled relative to the real vector space, simply because tensor products of real states can never sum up to give a $\sigma_y \otimes \sigma_y$ component.

In Sec. II we show that the entanglement of a two-rebit state is determined entirely by the $\sigma_y \otimes \sigma_y$ component of the state: a state ρ^{AB} is separable if and only if $\text{tr}(\rho^{AB} \sigma_y \otimes \sigma_y) = 0$. Furthermore, there is a *concurrence*, defined by

$$C(\rho^{AB}) \equiv |\text{tr}(\rho^{AB} \sigma_y \otimes \sigma_y)|, \quad (1.10)$$

which gives the two-rebit entanglement of formation in the same way that Wootters's concurrence gives the two-qubit entanglement of formation. In a concluding section (Sec. III), we discuss implications of our main result.

For the present, however, it should be noted that the expression in Eq. (1.10) *does not* correspond to Wootters' concurrence formula for qubits simply restricted to real density operators. This point is nicely illustrated by the real density operator

$$\rho^{AB} = \frac{1}{4} (I \otimes I + \sigma_y \otimes \sigma_y). \quad (1.11)$$

This state is a separable state relative to complex vector space, as can be checked by the partial transpose condition or by noting that it can be derived from an ensemble of two product states:

$$\rho^{AB} = \frac{1}{2} \left(\frac{1}{2}(I + \sigma_y) \otimes \frac{1}{2}(I + \sigma_y) + \frac{1}{2}(I - \sigma_y) \otimes \frac{1}{2}(I - \sigma_y) \right). \quad (1.12)$$

In this (eigen)decomposition, the density operator ρ^{AB} looks like it comes from the mixture of two spin states: both particles pointing in the $+y$ direction or both pointing in the $-y$ direction.

In contrast to this, the ensemble decomposition in Eq. (1.12) is not allowed relative to real vector space quantum mechanics. The real concurrence, Eq. (1.10), of this state is in fact $C = 1$, which means that the state is maximally entangled relative to real vector space. Perhaps more interestingly, this state is also a bound entangled state relative to the reals. This follows because it is separable relative to the complex numbers, i.e., it has no complex entanglement of formation, and hence, by the chain of inequalities in Eq. (1.7), it has no real distillable entanglement.

II. TWO-REBIT ENTANGLEMENT OF FORMATION

In this section we first review Wootters's spin-flip operation and how it leads to the real concurrence (1.10), and we then prove our main result, an explicit formula for the real entanglement of formation of an arbitrary state of two rebits.

The spin-flip operation for a single qubit is the anti-unitary operator $\mathcal{S} = i\sigma_y \mathcal{C}$, where \mathcal{C} denotes complex conjugation in the eigenbasis of σ_z . For a quantum state ρ of a bipartite system—we now drop the superscript AB to reduce clutter in the notation—the spin-flipped density operator, distinguished by a tilde, is

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y) \mathcal{C}(\rho) (\sigma_y \otimes \sigma_y). \quad (2.1)$$

The concurrence of a bipartite pure state $|\Psi\rangle$ is defined to be

$$C(\Psi) \equiv |\langle \Psi | \mathcal{S} | \Psi \rangle|. \quad (2.2)$$

Defining the concurrence of a mixed state ρ by

$$C(\rho) \equiv \min_{\{p_j, |\Psi_j\rangle\}} \sum_j p_j C(\Psi_j) , \quad (2.3)$$

where the minimum is taken over all possible ensemble decompositions of ρ , Wootters [5] showed that $C(\rho)$ is given by the explicit expression

$$C_W(\rho) = \max \left(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \right) , \quad (2.4)$$

where the λ_i are the (positive) eigenvalues of the operator $\rho \tilde{\rho}$ (or of the operator $\sqrt{\rho} \tilde{\rho} \sqrt{\rho}$) listed in order of decreasing magnitude.

For rebit states, the complex conjugation has no effect, and the spin flip simplifies to $i\sigma_y$, i.e., a 180° rotation about the y axis. The differences between concurrence and entanglement of formation for rebits and qubits can ultimately be traced to the fact that the spin flip for qubits is an anti-linear, as opposed to a linear operator. Hence, for a single-rebit pure state $|\psi\rangle$, the spin-flipped state is

$$|\tilde{\psi}\rangle \equiv i\sigma_y |\psi\rangle . \quad (2.5)$$

For a joint pure state $|\Psi\rangle$ of two rebits, we again define the *concurrence* to be the overlap between $|\Psi\rangle$ and the spin-flipped state $|\tilde{\Psi}\rangle = -\sigma_y \otimes \sigma_y |\Psi\rangle$, i.e.,

$$C(\Psi) \equiv |\langle \Psi | \tilde{\Psi} \rangle| = |\langle \Psi | \sigma_y \otimes \sigma_y | \Psi \rangle| , \quad (2.6)$$

and also define the concurrence of a mixed state according to Eq. (2.3).

The joint pure state $|\Psi\rangle$ can be written in terms of a Schmidt decomposition,

$$|\Psi\rangle = a_1 |e_1\rangle \otimes |f_1\rangle + a_2 |e_2\rangle \otimes |f_2\rangle , \quad (2.7)$$

where $|e_j\rangle$ and $|f_j\rangle$ are the (real) orthonormal eigenvectors of the marginal density operators for systems A and B , and a_1 and a_2 are the (positive) square roots of the corresponding eigenvalues. It is easy to verify that the concurrence of $|\Psi\rangle$ is $C(\Psi) = 2a_1a_2$. Thus, as noted by Wootters, the concurrence itself can serve as a measurement of entanglement, varying smoothly from 0 for product pure states to 1 for maximally entangled pure states. Indeed, the entanglement of formation of the pure state $|\Psi\rangle$ can be expressed in the form

$$E(\Psi) = H(a_1^2) = H\left(\frac{1 + \sqrt{1 - C^2}}{2}\right) \equiv \mathcal{E}[C(\Psi)] , \quad (2.8)$$

where

$$H(x) \equiv -x \log_2 x - (1 - x) \log_2 (1 - x) \quad (2.9)$$

is the binary Shannon entropy. The function $\mathcal{E}(C)$ is monotonically increasing and convex on the interval $0 \leq C \leq 1$.

Before proceeding to the problem at hand, let us delineate a few facts about ensemble decompositions of density operators in a real Hilbert space quantum mechanics. Consider a mixed state ρ , defined on a real Hilbert space of dimension d , whose eigendecomposition is

$$\rho = \sum_{j=1}^n \mu_j |\hat{e}_j\rangle \langle \hat{e}_j| = \sum_{j=1}^n |e_j\rangle \langle e_j|. \quad (2.10)$$

Here the vectors $|\hat{e}_j\rangle$ are the orthonormal eigenvectors of ρ , with corresponding eigenvalues μ_j , and the vectors $|e_j\rangle \equiv \sqrt{\mu_j} |\hat{e}_j\rangle = \rho^{1/2} |\hat{e}_j\rangle$ are subnormalized eigenvectors. In Eq. (2.10) the sum includes only eigenvectors with nonzero eigenvalues, $n \leq d$ thus being the rank of ρ . We can restrict attention to the support of ρ , the subspace of dimension n spanned by eigenvectors with nonzero eigenvalue. On this subspace ρ has a well defined inverse.

Consider now an arbitrary pure-state ensemble decomposition of ρ ,

$$\rho = \sum_{j=1}^m p_j |\hat{w}_j\rangle \langle \hat{w}_j| = \sum_{j=1}^m |w_j\rangle \langle w_j|. \quad (2.11)$$

The ensemble includes $m \geq n$ normalized vectors $|\hat{w}_j\rangle$, with probabilities p_j . The vectors $|w_j\rangle \equiv \sqrt{p_j} |\hat{w}_j\rangle$ are subnormalized, their lengths giving the probabilities. The real version of the pure-state decomposition theorem for density operators [17–19] says that a set of subnormalized vectors gives a decomposition of ρ if and only if the vectors can be written as

$$|w_j\rangle = \sum_{k=1}^n O_{kj} |e_k\rangle, \quad j = 1, \dots, m, \quad (2.12)$$

where O is an $n \times m$ matrix whose n rows are real m -dimensional orthonormal vectors. We can always extend O to be an $m \times m$ orthogonal matrix by adding additional rows.

Now notice that the projector onto the support of ρ can be written as

$$\Pi = \sum_{j=1}^m \rho^{-1/2} |w_j\rangle \langle w_j| \rho^{-1/2}, \quad (2.13)$$

where

$$\rho^{-1/2} |w_j\rangle = \sum_{k=1}^n O_{kj} |\hat{e}_k\rangle, \quad j = 1, \dots, m. \quad (2.14)$$

By adding additional orthonormal vectors $|\hat{e}_k\rangle$ for $k = n+1, \dots, m$, we can define m orthonormal vectors in an extended Hilbert space,

$$|\bar{w}_j\rangle \equiv \sum_{k=1}^m O_{kj} |\hat{e}_k\rangle, \quad j = 1, \dots, m. \quad (2.15)$$

Notice that

$$\Pi |\bar{w}_j\rangle = \rho^{-1/2} |w_j\rangle, \quad (2.16)$$

which implies that

$$|w_j\rangle = \rho^{1/2} |\bar{w}_j\rangle. \quad (2.17)$$

The extension of the resolution of Π in Eq. (2.13) to a set of orthonormal vectors in a higher-dimensional space is called a *Neumark extension* [20].

Now we apply these concepts to a two-rebit state ρ having the pure-state decomposition (2.11). The average concurrence of this decomposition satisfies the inequality

$$\langle C \rangle = \sum_j p_j C(\hat{w}_j) = \sum_j p_j |\langle \hat{w}_j | \sigma_y \otimes \sigma_y | \hat{w}_j \rangle| \geq \left| \sum_j p_j c(\hat{w}_j) \right| = |\langle c \rangle|, \quad (2.18)$$

where, following Wootters, we define the *preconcurrence* of the pure state $|\hat{w}_j\rangle$ without the absolute value signs that make the concurrence positive:

$$c(\hat{w}_j) \equiv \langle \hat{w}_j | \sigma_y \otimes \sigma_y | \hat{w}_j \rangle. \quad (2.19)$$

The attractive feature of the preconcurrence is that its average value is independent of the ensemble decomposition, being given by

$$\langle c \rangle = \sum_j \langle w_j | \sigma_y \otimes \sigma_y | w_j \rangle = \sum_j \langle \bar{w}_j | \rho^{1/2} (\sigma_y \otimes \sigma_y) \rho^{1/2} | \bar{w}_j \rangle = \text{tr}(\tau), \quad (2.20)$$

where

$$\tau \equiv \rho^{1/2} (\sigma_y \otimes \sigma_y) \rho^{1/2} \quad (2.21)$$

is a real, symmetric operator.

We now show that there is a pure-state ensemble whose average concurrence achieves the lower bound, $|\text{tr}(\tau)|$. We actually show something stronger, using the approach introduced by Wootters: there is a pure-state ensemble such that the preconcurrence of every member of the ensemble is $\text{tr}(\tau)$ and, hence, the concurrence of every member of the ensemble is $|\text{tr}(\tau)|$. This stronger result becomes important when we consider the entanglement of formation. To construct the desired ensemble, start with the eigendecomposition of ρ . If the eigendecomposition has only one member, then we are dealing with a pure state whose preconcurrence is $\text{tr}(\tau)$, and nothing further needs to be done. If the eigendecomposition has more than one member, consider the states with the largest and smallest preconcurrences. Since the average preconcurrence is $\text{tr}(\tau)$, the largest preconcurrence must be greater than or equal to $\text{tr}(\tau)$, and the smallest must be less than or equal to $\text{tr}(\tau)$. Consider the continuous sequence of two-dimensional orthogonal matrices that range from the identity matrix to the matrix that exchanges the subnormalized vectors for the states with the largest and smallest preconcurrences. Somewhere along this sequence, the state with the largest preconcurrence is transformed into one whose preconcurrence is $\text{tr}(\tau)$. Adopt the resulting ensemble decomposition. Now iterate this procedure, always choosing the states with the largest and smallest preconcurrences and transforming the state with the largest preconcurrence to one with preconcurrence equal to $\text{tr}(\tau)$. The result is an ensemble decomposition all of whose members have preconcurrence equal to $\text{tr}(\tau)$, as promised.

So far in this section we have shown that for a two-rebit density operator ρ , the minimum average concurrence over all the pure-state ensembles for ρ is

$$\langle C \rangle_{\min} \equiv \min_{\{p_j, |\hat{w}_j\rangle\}} \sum_j p_j C(\hat{w}_j) = |\text{tr}(\tau)| = |\text{tr}(\rho \sigma_y \otimes \sigma_y)|. \quad (2.22)$$

This justifies calling $|\text{tr}(\tau)|$ the concurrence of ρ , as in Eq. (1.10).

The entanglement of formation for a two-rebit state now follows with very little further work, since it satisfies the following chain of relations:

$$E(\rho) \equiv \min_{\{p_j, |\hat{w}_j\rangle\}} \sum_j p_j \mathcal{E}[C(\hat{w}_j)] \geq \min_{\{p_j, |\hat{w}_j\rangle\}} \mathcal{E}\left(\sum_j p_j C(\hat{w}_j)\right) = \mathcal{E}(\langle C \rangle_{\min}) = \mathcal{E}[|\text{tr}(\tau)|] . \quad (2.23)$$

The inequality follows from the convexity of the function $\mathcal{E}(C)$, and the immediately following equality follows from the monotonicity of $\mathcal{E}(C)$. To saturate the inequality requires an ensemble all of whose members have a concurrence equal to $\langle C \rangle_{\min} = |\text{tr}(\tau)|$. Having just constructed an ensemble, we conclude that the entanglement of formation of a two-rebit state ρ is given by

$$E(\rho) = \mathcal{E}(|\text{tr}(\tau)|) = \mathcal{E}[C(\rho)] . \quad (2.24)$$

III. DISCUSSION

To conclude, we reiterate that we now possess a complete expression for the entanglement of formation of two rebits:

$$E(\rho^{AB}) = H\left(\frac{1 + \sqrt{1 - C^2(\rho^{AB})}}{2}\right) , \quad (3.1)$$

where

$$C(\rho^{AB}) = |\text{tr}(\tau)| = |\text{tr}(\rho^{AB} \sigma_y \otimes \sigma_y)| . \quad (3.2)$$

In particular, this expression implies that ρ^{AB} is separable relative to real vector space if and only if $\text{tr}(\rho^{AB} \sigma_y \otimes \sigma_y) = 0$. Notice that this separability condition is equivalent to saying that ρ^{AB} is real separable if and only if it is unchanged by partial transposition, i.e.,

$$\rho^{AB} = (\rho^{AB})^{T_A} = (\rho^{AB})^{T_B} , \quad (3.3)$$

where T_A (T_B) denotes partial transposition of system A (B) in any orthonormal basis.

It is worth stressing the difference between the expression (3.2) for the concurrence in a real vector space and the Wootters formula (2.4) for concurrence in standard quantum theory. If we let ν_j be the eigenvalues of τ , ranked in order of decreasing absolute value, the real concurrence is

$$C(\rho^{AB}) = |\nu_1 + \nu_2 + \nu_3 + \nu_4| . \quad (3.4)$$

In contrast, for the Wootters concurrence, one first finds the eigenvalues of $\tau\tau^\dagger = \tau^2$, these being given by $\lambda_j = \nu_j^2$; then the Wootters concurrence is

$$C_W(\rho^{AB}) = \max(0, |\nu_1| - |\nu_2| - |\nu_3| - |\nu_4|) \leq C(\rho^{AB}) . \quad (3.5)$$

This difference is illustrated by the class of real states of the form

$$\rho^{AB} = \frac{1}{4} \left(I \otimes I + \alpha (\sigma_y \otimes \sigma_y) \right), \quad (3.6)$$

where α is a positive real number that ranges from 0 to 1. For these states, the operator τ , given by

$$\tau = \frac{1}{4} \left(\alpha (I \otimes I) + \sigma_y \otimes \sigma_y \right), \quad (3.7)$$

has doubly degenerate eigenvalues $\frac{1}{4}(\alpha \pm 1)$. The real concurrence is $C = \alpha$, whereas the Wootters concurrence is $C_W = 0$. Thus these states are complex separable, but real entangled, except for $\alpha = 0$. Moreover, because these states are complex separable, they have no distillable entanglement, so their real entanglement is bound.

Our results can be considered a first step toward getting a better understanding of which features of quantum entanglement are unique to standard quantum mechanics and which are more generic across various foil theories. As just shown, the states (3.6) show that bound entanglement is sometimes nothing more than separability with respect to a larger field (in this case the complex numbers of standard quantum mechanics). One might ask to what extent this is true of bound entanglement in standard quantum mechanics. How many bound entangled states in standard quantum mechanics are bound because they are separable with respect to a quaternionic theory [7]?

Another interesting fact is how the regions of entangled vs. separable states within the full set of quantum states differ in going from real to complex quantum mechanics. In the complex theory, the maximally mixed state $\rho = \frac{1}{4}I_4$ of two qubits is surrounded by an open set of separable states [21,22]. In the real theory, however, the states (3.6) demonstrate that there are entangled states arbitrarily close to the maximally mixed state.

Where all this will lead, we are not quite sure, but in general it forms part of a larger effort to understand the nature of entanglement in our quantum world.

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